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The G-strong containment for locally defined formations of finite soluble groups ☆

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Abstract

Let $\mathfrak{F}$ and $\mathfrak{H}$ be two saturated formations of finite soluble groups, with canonical definition F and H, respectively, and $\mathrm{Char}(\mathfrak{F}) \subseteq \mathrm{Char}(\mathfrak{H})$. We say that $\mathfrak{F}$ is G-strong containment in $\mathfrak{H}$, when for all $p \in \mathrm{Char}(\mathfrak{F})$ and for all $H \in \mathfrak{H}$ the $H(p)$-Residual of H is in a $\mathfrak{F}$-projector E of H containment. In this case we write $\mathfrak{F} \ll_{\mathsf{G}} \mathfrak{H}$. In this work we wants to show a characterization of the saturated formation $\mathfrak{H}$, which contain G-strongly the formations $\mathfrak{S}_\pi$, $\mathfrak{N}\mathfrak{X}$, $\mathfrak{N}^k$ and $\ddot{\mathfrak{U}}$, respectively.

1. Preliminaries

All groups considered in this paper are finite and soluble. We refer to [5] for the notation and basic results on classes of groups.

Let $\mathfrak{F}$ and $\mathfrak{H}$ be two saturated formations with $\mathfrak{F} \subseteq \mathfrak{H}$. The fact that $\mathfrak{F}$ is in $\mathfrak{H}$ contained, does not imply a corresponding inclusion between their projectors. For example, in the symmetric group of order 4, an $\ddot{\mathfrak{U}}$-projector cannot be contained in an $\mathfrak{N}\mathfrak{A}$-projector, although, $\ddot{\mathfrak{U}} \subseteq \mathfrak{N}\mathfrak{A}$.

Cline [1] was the first to consider inclusion between projectors to define a new relation between classes, namely the strong containment in the context of the family $\mathcal{F}$ of saturated formations in $\mathfrak{S}$.

☆ It is a part of the PhD thesis of the author, see [I. Gutiérrez, Zur starken Enthaltenseinsrelation für gesättigte Formationen auflösbarer Gruppen, Dissertation, Johannes Gutenberg Universität-Mainz, 2002. [6]].

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The strong containment for Schunck classes is quite well understood, see, for example, [5, VI]. But in $\mathcal{F}$ it is not so well understood see, for example, Cline [1] and D'Arcy [2].

1.1. Definition. Let $\mathfrak{F}$ and $\mathfrak{H}$ be two saturated formations. We say that $\mathfrak{F}$ is strongly contained in $\mathfrak{H}$ if, in each $G \in \mathfrak{S}$ an $\mathfrak{F}$-projector of G is contained (as subgroup) in an $\mathfrak{H}$-projector of G. We write $\mathfrak{F} \ll \mathfrak{H}$. It is clear that $(\mathcal{F}, \ll)$ is a partial order.

1.2. Definition. Let $\mathfrak{F}$ and $\mathfrak{H}$ are two saturated formations. Define the class $(\mathfrak{F} \downarrow \mathfrak{H})$ as follows:

$$(\mathfrak{F} \downarrow \mathfrak{H}) := \bigl(G \in \mathfrak{S}\colon \operatorname{Proj}_{\mathfrak{F}}(G) \subseteq \mathfrak{H}\bigr).$$

This class is a formation. See [5, IV (1.2)].

1.3. Satz. (Doerk [3]) Let $\mathfrak{F}$ a saturated formation with canonical local definition F and for all $p \in \mathbb{P}$ let $f^*(p) := (\mathfrak{F} \downarrow F(p))$. Then

(1) The formation function f^* is a full local definition of $\mathfrak{F}$.
(2) The f^*-central chief factors of a group G are precisely those chief factors which are covered by an $\mathfrak{F}$-projector of G.

Proof. See [5, IV (5.19)]. $\square$

Now we introduce a result which will be needed later, namely we describe the saturated formation $\mathfrak{H}$ which contain strongly the formation of the π-groups.

1.4. Lemma. *Let H be the canonical local definition of a saturated formation $\mathfrak{H}$ and let $\pi \subseteq$* $\operatorname{Char}(\mathfrak{H})$. *Then any two of the following statements are equivalent*:

(1) $\mathfrak{S}_\pi \ll \mathfrak{H}$ (*i.e. in each group G the $\mathfrak{H}$-projectors have π'-index*);
(2) *$H(q) \subseteq H(p)$ for all $p \in \pi$ and for all $q \in \mathbb{P}$*;
(3) *$H(p) = \mathfrak{H}$ for all $p \in \pi$*;
(4) *$h^*(p) = \mathfrak{S}$ for all $p \in \pi$*;
(5) $\mathfrak{S}_\pi \mathfrak{H} = \mathfrak{H}$.

Proof. See [5, IV (5.20)]. $\square$

1.5. Lemma. (D'Arcy [2]) *Let $\mathfrak{F} = LF(F) \subseteq \mathfrak{H} = LF(H)$, where F and H are the canonical definitions. Then $\mathfrak{F} \ll \mathfrak{H}$ if and only if for each $H \in \mathfrak{H}$ an $\mathfrak{F}$-projector E of H satisfies $H^{H(p)} \leqslant E^{F(p)}$ for each $p \in$* $\operatorname{Char}(\mathfrak{F})$.

Proof. See [5, VII (5.1)]. $\square$

Now we want to use this criterion to introduce a new concept, namely the G-strong containment of saturated formations of finite soluble groups. We change it a little to get a weaker definition.

1.6. Definition. Let $\mathfrak{F}$ and $\mathfrak{H} = LF(H)$ be two saturated formations with H the canonical definition and $\mathrm{Char}(\mathfrak{F}) \subseteq \mathrm{Char}(\mathfrak{H})$. We say, that $\mathfrak{F}$ is G-strongly contained in $\mathfrak{H}$ if, for each $H \in \mathfrak{H}$ an $\mathfrak{F}$-projector E von H satisfies the property $H^{H(p)} \leqslant E$ for each $p \in \mathrm{Char}(\mathfrak{F})$. We write $\mathfrak{F} \ll_{\mathsf{G}} \mathfrak{H}$.

Diagram:

$$\begin{tikzcd} H \arrow[d, dash] \\ E \arrow[d, dash] \\ H^{H(p)} \arrow[d, dash] \\ \langle 1 \rangle \end{tikzcd}$$

1.7. Remark. Let $\mathfrak{F}$ and $\mathfrak{H} = LF(H)$ be two saturated formations.

(a) It follows immediately from the last definition that $\mathfrak{F} \ll_{\mathsf{G}} \mathfrak{H}$ whenever $\mathfrak{F} \ll \mathfrak{H}$. But the reciprocal does not hold in general.
(b) If $\mathfrak{H} \subseteq \mathfrak{F}$, then $\mathfrak{F} \ll_{\mathsf{G}} \mathfrak{H}$. In fact, let $H \in \mathfrak{H}$. Then is H an element of $\mathfrak{F}$ and therefore $H \in \mathrm{Proj}_{\mathfrak{F}}(H)$. Es is clear, that $H^{H(r)} \leqslant H$ for all $r \in \mathrm{Char}(\mathfrak{F})$ hold. This proves that $\mathfrak{F} \ll_{\mathsf{G}} \mathfrak{H}$. Since any finite nilpotent group is supersoluble, we have $\ddot{\mathfrak{U}} \ll_{\mathsf{G}} \mathfrak{N}$. But note that $\ddot{\mathfrak{U}} \ll \mathfrak{N}$ does not hold.
(c) Let $\mathfrak{N}_\pi$ be the class all nilpotente π-groups. The canonical local definition H of the formation $\mathfrak{N}_\pi$ is given by:

$$H(r) = \begin{cases} \mathfrak{S}_r & \text{falls } r \in \pi, \\ \emptyset & \text{falls } r \notin \pi. \end{cases}$$

Let $H \in \mathfrak{N}_\pi$ and $E \in \mathrm{Hall}_\pi(H)$, then is $H^{H(r)} = H^{\mathfrak{S}_r} \leqslant E$ for all $r \in \pi$. Also hold $\mathfrak{S}_\pi \ll_{\mathsf{G}} \mathfrak{N}_\pi$. Moreover, $\mathfrak{N}_\pi \neq \mathfrak{S}_\pi \mathfrak{N}_\pi$, implying that the class $\mathfrak{S}_\pi$ is not contained in $\mathfrak{N}_\pi$.

Now we want to characterize the locally defined formation $\mathfrak{H}$, which contain G-strongly the formation $\mathfrak{S}_\pi$.

1.8. Lemma. *Let $\mathfrak{F}$ be a saturated formation with $\mathsf{s}\mathfrak{F} = \mathfrak{F}$, i.e. $\mathfrak{F}$ is closed under subgroups. Let $\mathfrak{H} = LF(H)$ with H the canonical local definition, and assume that $\varrho := \mathrm{Char}(\mathfrak{F}) \subseteq \mathrm{Char}(\mathfrak{H})$. If $\mathfrak{F} \ll_{\mathsf{G}} \mathfrak{H}$, then is*

$$\mathfrak{H} \subseteq \bigcap_{r \in \varrho} \mathfrak{F} H(r).$$

Proof. Let $H \in \mathfrak{H}$. Then is $H^{H(r)} \leqslant E$ for all $r \in \varrho$, where E is an $\mathfrak{F}$-projector of H. Since $\mathsf{s}\mathfrak{F} = \mathfrak{F}$ and $E \in \mathfrak{F}$, is $H^{H(r)} \in \mathfrak{F}$ for all $r \in \varrho$. Then hold $H \in \mathfrak{F} H(r)$ for all $r \in \varrho$. It follows $H \in \bigcap_{r \in \varrho} \mathfrak{F} H(r)$. $\square$

We consider now Lemma 1.8 with $\mathfrak{F} = \mathfrak{S}_\pi$. We will prove that in this case hold the reciprocal.

1.9. Lemma. *Let* $\mathfrak{H} = LF(H)$ *with* $\pi \subseteq \mathrm{Char}(\mathfrak{H})$. $\mathfrak{S}_\pi \ll_{\mathsf{G}} \mathfrak{H}$ *if and only if*

$$\mathfrak{H} \subseteq \bigcap_{r\in\pi} \mathfrak{S}_\pi H(r).$$

Proof. First suppose that $\mathfrak{H} \subseteq \bigcap_{r\in\pi} \mathfrak{S}_\pi H(r)$. Let $H \in \mathfrak{H}$, then $H \in \mathfrak{S}_\pi H(r)$ for all $r \in \pi$. It follows that the $H(r)$-residual of H is a π-group for all $r \in \pi$. Let $E \in \mathrm{Hall}_\pi(H)$. Then hold: $H^{H(r)} \leqslant E$ for all $r \in \pi$ and this proves that $\mathfrak{S}_\pi \ll_{\mathsf{G}} \mathfrak{H}$. The sufficiency follow from 1.8. □

1.10. Remark. Let $\mathfrak{H} = LF(H)$ with $\pi \subseteq \mathrm{Char}(\mathfrak{H})$. Define $\mathfrak{X} := \bigcap_{r\in\pi} H(r)$. Then

$$\bigcap_{r\in\pi} \mathfrak{S}_\pi H(r) = \mathfrak{S}_\pi \mathfrak{X}.$$

Proof.

(1) It is clear that $\mathfrak{X} \subseteq H(r)$ for all $r \in \pi$. Then $\mathfrak{S}_\pi \mathfrak{X} \subseteq \mathfrak{S}_\pi H(r)$ for all $r \in \pi$. It follows $\mathfrak{S}_\pi \mathfrak{X} \subseteq \bigcap_{r\in\pi} \mathfrak{S}_\pi H(r)$.
(2) Let $H \in \bigcap_{r\in\pi} \mathfrak{S}_\pi H(r)$. Then is $H^{H(r)}$ for all $r \in \pi$ a normal π-subgroup of H and therefore is $H^{H(r)}$ contained in $O_\pi(H)$ for all $r \in \pi$. It follows: $H/O_{\pi(H)} \in H(r)$ for all $r \in \pi$. We have $H/O_{\pi(H)} \in \mathfrak{X}$ and this proves that $H \in \mathfrak{S}_\pi \mathfrak{X}$. □

1.11. Theorem. *Let* $\mathfrak{H}$ *be a saturated formation and* $\pi \subseteq \mathrm{Char}(\mathfrak{H})$. $\mathfrak{S}_\pi \ll_{\mathsf{G}} \mathfrak{H}$ *if and only if there exist a formation* $\mathfrak{X}$ *such that* $\mathfrak{H} \subseteq \mathfrak{S}_\pi \mathfrak{X}$.

Proof. This follows from 1.9 and 1.10. □

The following theorem will prove that in the case $\pi = \{p\}$ the concepts strong containment and G-strong containment are equivalent, if p belongs to the characteristic of $\mathfrak{H}$.

1.12. Theorem. *Let* H *be the canonical local definition of a saturated formation* $\mathfrak{H}$ *and let* $p \in \mathrm{Char}(\mathfrak{H})$. *Then any two of the following statements are equivalent*:

(1) $\mathfrak{S}_p \ll \mathfrak{H}$;
(2) $H(q) \subseteq H(p)$ *for all* $q \in \mathbb{P}$;
(3) $H(p) = \mathfrak{H}$;
(4) $h^*(p) = \mathfrak{S}$;
(5) $\mathfrak{S}_p\, \mathfrak{H} = \mathfrak{H}$;
(6) $\mathfrak{S}_p \ll_{\mathsf{G}} \mathfrak{H}$.

Proof.

$$\begin{aligned} \mathfrak{S}_p \ll_{\mathsf{G}} \mathfrak{H} &\underset{(1.9)}{\Leftrightarrow} \mathfrak{H} \subseteq \mathfrak{S}_p\, H(p) \\ &\Leftrightarrow \mathfrak{H} \subseteq H(p) \end{aligned}$$

$$\Leftrightarrow \mathfrak{H} = H(p)$$

$$\underset{(1.4)}{\Leftrightarrow} \mathfrak{S}_p \ll \mathfrak{H}$$

and hold the equivalency of (1) to (6). □

2. The formations $\mathfrak{N}$, $\mathfrak{N}\mathfrak{X}$, $\mathfrak{N}^k$ and $\ddot{\mathfrak{U}}$

The formation $\mathfrak{N}$

We recall first the definition of the upper nilpotent series $\{F_i(G)\}_{i\geqslant 0}$ of a group G.

2.1. Definition. Let G be a soluble group. The upper nilpotent series of G is defined recursively by:

(1) It is $F_0(G) = \langle 1\rangle$.
(2) For $i \geqslant 1$ let $F_i(G)/F_{i-1}(G) = F(G/F_{i-1}(G))$.

The smallest integer l such that $F_l(G) = G$ is called the Fitting length (or nilpotent length) of G and is denoted by $l(G)$.

2.2. Theorem. *If* $\mathfrak{N} \ll_{\mathsf{G}} \mathfrak{H} = LF(H)$*, then exist a formation* $\mathfrak{X}$ *such that* $\mathfrak{H}$ *has the form* $\mathfrak{H} = \mathfrak{N}\mathfrak{X}$.

Proof. First suppose that $\mathfrak{N} \ll_{\mathsf{G}} \mathfrak{H}$ and let $H \in \mathfrak{H}$. Then the residual $H^{H(r)}$ for all $r \in \mathbb{P}$ is contained in a Carter subgroup E of H and therefore for each prime r hold: $H^{H(r)} \leqslant F(H)$. Consequently $H/F(H) \in \bigcap_{r\in\mathbb{P}} H(r)$. Define $\mathfrak{X} := \bigcap_{r\in\mathbb{P}} H(r)$. Then hold $H \in \mathfrak{N}\mathfrak{X}$ and we have $\mathfrak{H} \subseteq \mathfrak{N}\mathfrak{X}$.

Conversely, define $\mathfrak{Y} = \mathfrak{N}\mathfrak{X}$. Then is $\mathfrak{Y} = LF(Y)$, where $Y(p) = \mathfrak{S}_p\mathfrak{X}$ for all prime p. We have: $Y(p) = \mathfrak{S}_p\mathfrak{X} \subseteq \mathfrak{S}_p H(p) = H(p)$ for all p. Then is $\mathfrak{N}\mathfrak{X} = \mathfrak{Y} \subseteq \mathfrak{H}$. □

2.3. Remark. The reciprocal of the last theorem does not hold in general. For example, from $\mathfrak{H} = \mathfrak{N}\mathfrak{A}$ does not follow that $\mathfrak{N} \ll_{\mathsf{G}} \mathfrak{H}$.

Proof. Let A_4 the alternating group on 4 letters. It is clear that $A_4 \in \mathfrak{N}\mathfrak{A}$. The 3-Sylow subgroups of A_4 are his $\mathfrak{N}$-projectors and es hold $A_4^{H(5)} = V_4$, the Klein's 4-group. If $E \in \mathrm{Syl}_3(A_4)$, then it is clear that $V_4 \nleqslant E$. Consequently does not hold that $\mathfrak{N} \ll_{\mathsf{G}} \mathfrak{H}$. □

The formation $\mathfrak{N}\mathfrak{X}$

Cline has classified the formations $\mathfrak{H}$, which contain strongly the formation $\mathfrak{N}\mathfrak{X}$, where $\mathfrak{X}$ is some formation.

Now we want to consider this problem under the new definition, but we suppose that the class $\mathfrak{X}$ is a Fitting formation.

First we introduce the concept of an $\mathfrak{F}$-radical, where $\mathfrak{F}$ is a Fitting class. It is the dual concept of an $\mathfrak{F}$-residual, when $\mathfrak{F}$ is a formation; its definitions appears in next lemma.

2.4. Lemma. *Let $\mathfrak{X}$ be a N_0-closed class and G a finite group. Then the set*

$$\mathcal{S} = \{N \mid N \trianglelefteq\trianglelefteq G \text{ and } N \in \mathfrak{X}\},$$

partially ordered by inclusion, has a unique maximal element, denoted by $G_{\mathfrak{X}}$, and if $\mathfrak{X}$ is a Fitting class and $K \trianglelefteq\trianglelefteq G$, then $K_{\mathfrak{X}} = K \cap G_{\mathfrak{X}}$.

Proof. See [5, II (2.9)]. $\square$

2.5. Theorem. *Let $\mathfrak{X}$ be a Fitting formation. If $\mathfrak{F} = \mathfrak{N}\mathfrak{X} \ll_{\mathsf{G}} \mathfrak{H} = LF(H)$, then there exist a formation $\mathfrak{Y}$, such that $\mathfrak{H} \subseteq \mathfrak{N}\mathfrak{Y}$.*

Proof. Let $H \in \mathfrak{H}$. Then hold $H^{H(r)} \leqslant E$ for all $r \in \mathbb{P}$, where $E \in \mathrm{Proj}_{\mathfrak{F}}(H)$. We have $H^{H(r)} \leqslant H_{\mathfrak{N}\mathfrak{X}}$ for all $r \in \mathbb{P}$. Define

$$\mathfrak{Z} := \bigcap_{r \in \mathbb{P}} H(r).$$

Then $H/H_{\mathfrak{N}\mathfrak{X}} \in \mathfrak{Z}$ and therefore $H/F(H) \in \mathfrak{X}\mathfrak{Z}$. Consequently $H \in \mathfrak{N}\mathfrak{X}\mathfrak{Z}$. Define now $\mathfrak{Y} := \mathfrak{X}\mathfrak{Z}$. This proves that $\mathfrak{H} \subseteq \mathfrak{N}\mathfrak{Y}$. $\square$

The formation $\mathfrak{N}^k$

If $k \in \mathbb{N}$, let $\mathfrak{N}^k = \mathfrak{N} \circ \cdots \circ \mathfrak{N}$ (k copies), the class of soluble groups of nilpotent length at most k. It is well known, that the formation $\mathfrak{F} = \mathfrak{N}^k$ has the canonical local definition $F(p) = \mathfrak{S}_p\mathfrak{N}^{k-1}$ for all $p \in \mathbb{P}$. (The symbol $\mathfrak{N}^0$ is to be interpreted as the class of groups of order 1.)

In the following theorem we look a property of the saturated formation $\mathfrak{H}$ which contains G-strongly the formation $\mathfrak{N}^k$.

2.6. Theorem. *Let $\mathfrak{H} = LF(H)$. If $\mathfrak{N}^k \ll_{\mathsf{G}} \mathfrak{H}$, then there exist a formation $\mathfrak{X}$, such that $\mathfrak{H} \subseteq \mathfrak{N}^k\mathfrak{X}$.*

Proof. Suppose $\mathfrak{N}^k \ll_{\mathsf{G}} \mathfrak{H}$ and let $H \in \mathfrak{H}$. Then $H^{H(r)} \leqslant E$ for all $r \in \mathbb{P}$, where E is an $\mathfrak{N}^k$-projector of H. Therefore is the residual $H^{H(r)}$ for all $r \in \mathbb{P}$ contained in $F_k(H)$. Consequently:

$$H/F_k(H) \in \bigcap_{r \in \mathbb{P}} H(r).$$

Define $\mathfrak{X} := \bigcap_{r \in \mathbb{P}} H(r)$. Then $H \in \mathfrak{N}^k\mathfrak{X}$. $\square$

The formation $\ddot{\mathfrak{U}}$

Let $\mathfrak{A}(p-1) = \mathfrak{A} \cap \mathfrak{E}(p-1)$ the class of finite abelian groups of exponent dividing $p-1$. Let $u(p) = \mathfrak{A}(p-1)$ for all $p \in \mathbb{P}$. The saturated formation $\ddot{\mathfrak{U}} = LF(u)$ is the class of supersoluble groups.

The next theorems provides properties of the formations $\ddot{\mathfrak{U}}$ and $\mathfrak{N}^2$, which are used to characterizer the saturated formation $\mathfrak{H}$, which contains G-strongly the formation $\ddot{\mathfrak{U}}$.

2.7. Theorem. *If $\mathfrak{F} = LF(f) \subseteq \mathfrak{N}^2$, then $\mathfrak{F} = \mathsf{s}\mathfrak{F}$.*

Proof. See [5, IV (3.18)]. □

2.8. Theorem. *The saturated formation $\ddot{\mathfrak{U}}$ is closed under forming subgroups and $\ddot{\mathfrak{U}} \subseteq \mathfrak{N}\mathfrak{A}$.*

Proof. See [5, VII (2.1)]. □

2.9. Theorem. *Let $\mathfrak{H} = LF(H)$. If $\ddot{\mathfrak{U}} \ll_{\mathsf{G}} \mathfrak{H}$, then there exists a formation $\mathfrak{X}$, such that $\mathfrak{H} \subseteq \mathfrak{N}^2\mathfrak{X}$.*

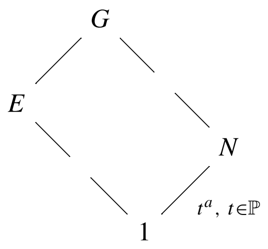
Proof. Let $H \in \mathfrak{H}$ and let E be an $\ddot{\mathfrak{U}}$ -projector of H. Then $H^{H(r)} \leqslant E$ for all $r \in \mathbb{P}$. From 2.8 hold $E \in \mathfrak{N}^2$ and from Theorem 2.7 is the formation $\mathfrak{N}^2$ closed under forming subgroups. Consequently $H^{H(r)} \leqslant F_2(H)$ for all $r \in \mathbb{P}$. Then we have $H/F_2(H) \in H(r)$ for all $r \in \mathbb{P}$. Define

$$\mathfrak{X} := \bigcap_{r \in \mathbb{P}} H(r).$$

Finally we have $H \in \mathfrak{N}^2\mathfrak{X}$ and therefore $\mathfrak{H} \subseteq \mathfrak{N}^2\mathfrak{X}$. □

2.10. Theorem. *Let $\mathfrak{H}$ be a saturated formation with $LF(\ddot{u}) = \ddot{\mathfrak{U}} \ll_{\mathsf{G}} \mathfrak{H} = LF(H)$ and $\mathfrak{H} \nsubseteq \ddot{\mathfrak{U}}$. Then contain $\mathfrak{H}$ groups of arbitrary nilpotent length.*

Proof. Let $G \in \mathfrak{H} \setminus \ddot{\mathfrak{U}}$ a group of minimal order. Then G is a primitive group (see, for example, [5, II (2.5)]). We have then the following situation:



where E is an $\ddot{\mathfrak{U}}$-projector of G and N is a minimal normal subgroup of G. From hypothesis hold that $G^{H(p)} \leqslant E$ for all $p \in \mathbb{P}$. Because G is a primitive group, we have $G \in H(p)$ for all $p \in \mathbb{P}$.

$E \notin \ddot{u}(t)$. It has $|E|$ a divisor $r > 2$: If E is an abelian 2-group, whose exponent divide $t - 1$, then G is supersoluble and that is not possible. Also is E a 2-group, whose exponent do not divide $t - 1$. In particular $4 \mid \exp(E)$.

Also hold:

It is r a prim divisor of $|E|$, then is $r > 2$ or E hat an element with order 4. It is $r > 2$, then there exists from the famous Dirichlet's theorem infinity prim numbers q of the form $q = nr + 2$, where $n \in \mathbb{N}$ and furthermore $q \nmid |G|$. See, for example, [7, p. 108].

It is $x \in E$ an element of order 4, then there exists infinite prim numbers q of the form $q = 4n + 3$, where $n \in \mathbb{N}$ and furthermore $q \nmid |G|$.

Let now M_1 be an faithful irreducible module of G over $GF(q_1)$. Define $G_1 := [M_1]G$ (the semidirect product of M_1 with G) and let E_1 be an $\ddot{\mathfrak{U}}$-projector of G_1. Then hold $M_1 \nleqslant E_1$ and by that follow $G_1 \in H(p)$ for all $p \in \mathbb{P}$.

With the same argumentation it follows the assertion. □

3. Further results

Let $\mathfrak{H}$ and $\mathfrak{F}$ be two saturated formations with canonical definition H respectively F. In general we have $H(p) \nsubseteq f^*(p)$, where $p \in \mathbb{P}$, see, for example, [4, (2.7)]. We want to find necessary conditions for $H(p) \subseteq f^*(p)$ with $p \in \mathbb{P}$ fixed under the condition $LF(F) = \mathfrak{F} \ll_{\mathsf{G}} \mathfrak{H} = LF(H)$.

3.1. Theorem. *Let $LF(F) = \mathfrak{F} \ll_{\mathsf{G}} \mathfrak{H} = LF(H)$ and $p \in \varrho := \mathrm{Char}(\mathfrak{F})$. It is $H(p) \nsubseteq f^*(p)$, then is $H(p) = \bigcap_{r \in \varrho} H(r)$.*

Proof. Define $\mathfrak{X}_p := H(p) \setminus f^*(p)$. First we will prove that $\mathfrak{X}_p \subseteq H(r)$ for all $r \in \varrho$. Let $H \in \mathfrak{X}_p$ and define $G := C_p \wr_{\mathrm{reg}} H = BH$, where B is the base group of G. Then is $G \in \mathfrak{S}_p H(p) = H(p) \subseteq \mathfrak{H}$.

It is E an $\mathfrak{F}$-projector of H, then is $E^* := EC_B(E^{F(p)})$ an $\mathfrak{F}$-projector of G (see, for example, [5, IV (5.16)(a)]).

From hypothesis we have $\mathfrak{F} \ll_{\mathsf{G}} \mathfrak{H}$ and therefore $G^{H(r)} \leqslant E^*$ for all $r \in \varrho$. By [5, IV (5.16)(a)] the residual $G^{H(r)}$ contains $H^{H(r)}$ and hence its normal closure $[B, H^{H(r)}]H^{H(r)}$. Then hold:

$$\left[B, H^{H(r)}\right] \leqslant G^{H(r)} \leqslant E\, C_B\left(E^{F(p)}\right).$$

Clearly is $[B, H^{H(r)}] \leqslant B$. With the Dedekind identity follow:

$$\left[B, H^{H(r)}\right] \leqslant B \cap E\, C_B\left(E^{F(p)}\right) = C_B\left(E^{F(p)}\right).$$

By [5, B (11.1)(e)] one the following three cases arises:

(1) $H^{H(r)} = \langle 1 \rangle$, or
(2) $E^{F(p)} = \langle 1 \rangle$, or
(3) $|E^{F(p)}| = p = 2$.

If $E^{F(p)} = \langle 1 \rangle$, then $E \in F(p)$ and $H \in f^*(p)$, contrary to the choice of H.

If $|E^{F(p)}| = p = 2$, then $E \in \mathfrak{S}_2 F(2) = F(2)$, and $H \in f^*(2)$, a contradiction.

The only remaining possibility is that $H^{H(r)} = \langle 1 \rangle$, and this means that $H \in H(r)$ for all $r \in \varrho$. This proves that $\mathfrak{X}_p \subseteq H(r)$ for all $r \in \varrho$.

Let now $H \in \mathfrak{X}_p$, where $K \in H(p)$ is arbitrary. Then $(H \times K) \in \mathfrak{X}_p$, and follow $K \in \mathsf{Q}(H \times K) \subseteq \mathsf{Q}\mathfrak{X}_p \subseteq \mathsf{Q}H(r) = H(r)$ for all $r \in \varrho$.

Also hold

$$H(p) \subseteq \bigcap_{r \in \varrho} H(r).$$

The reciprocal inclusion is clear. Then follow the assertion. $\square$

3.2. Corollary. *Let $\mathfrak{H}$ and $\mathfrak{F}$ be two saturated formations with canonical definition H respectively F. Let $\mathfrak{N} \subseteq \mathfrak{F} \ll_{\mathsf{G}} \mathfrak{H}$. Then for each $p \in \mathbb{P}$ either $H(p) \subseteq f^*(p)$ or $h^*(p) = H(p)$.*

Proof. Suppose that $H(p) \nsubseteq f^*(p)$ for some $p \in \mathbb{P}$. Then is from 3.1 $H(p) \subseteq H(q)$ for all $q \in \mathbb{P}$. Let $G \in h^*(p)$ and let H be an $\mathfrak{H}$-projector of G. Then is $H \in H(p)$ and $H \in H(q)$ for all $q \in \mathbb{P}$. From [5, IV (5.21)] follow $G = H \in H(p)$. This proves that $h^*(p) \subseteq H(p)$. The rest is trivial. $\square$

Now we consider a new variation of the concept strong containment for saturated formations. We define the D-strong containment relation.

3.3. Definition. Let $\mathfrak{F}$ and $\mathfrak{H} = LF(H)$ be two saturated formations with H the canonical definition and $\mathrm{Char}(\mathfrak{F}) \subseteq \mathrm{Char}(\mathfrak{H})$. We say, that $\mathfrak{F}$ is D-strongly contained in $\mathfrak{H}$ if, for each $H \in \mathfrak{H}$ an $\mathfrak{F}$-projector E von H satisfies the property $H^{H(p)} \leqslant E$ for each $p \in \mathrm{Char}(\mathfrak{H})$. We write $\mathfrak{F} \ll_{\mathsf{D}} \mathfrak{H}$.

From Remark 1.7(c) follow that $\mathfrak{S}_\pi \ll_{\mathsf{D}} \mathfrak{N}_\pi$. We have proved in 1.12 that "$\mathfrak{S}_p \ll \mathfrak{H}$" is equivalent to "$\mathfrak{S}_p \ll_{\mathsf{G}} \mathfrak{H}$," when $p \in \mathrm{Char}(\mathfrak{H})$. Now we occupy with the case $p \notin \mathrm{Char}(\mathfrak{H})$ with reference to the D-strong relation.

3.4. Lemma. *Let $\mathfrak{H} = LF(H)$ be a saturated formation and let p be a prim number with $p \notin \pi := \mathrm{Char}(\mathfrak{H})$. Is $\mathfrak{S}_p \ll_{\mathsf{D}} \mathfrak{H}$, then is $\mathfrak{H} = \mathfrak{S}_\pi$.*

Proof. Let $H \in \mathfrak{H}$. Since $p \notin \mathrm{Char}(\mathfrak{H})$, is $H(p) = \emptyset$. Therefore is H a p'-group. Suppose that $\mathfrak{S}_p \ll_{\mathsf{D}} \mathfrak{H}$. Then is $H^{H(r)} \leqslant E$ for all $r \in \pi$, where $E \in \mathrm{Syl}_p(H)$. Since too $H/H^{H(r)}$ is a p'-group, we have $H^{H(r)} = E$.

Consequently is $H^{H(r)}$ a Sylow p-normal subgroup of H. Then E must be equal $\langle 1 \rangle$ and follow that $H \in H(r)$ for all $r \in \pi$. This proves that $\mathfrak{H} \subseteq H(r)$ for all $r \in \pi$. But it holds too that $H(r) \subseteq \mathfrak{H}$ for all $r \in \pi$. Therefore is $\mathfrak{H} = H(r)$ for all $r \in \pi$ and $\mathfrak{H} = \mathfrak{S}_\pi$. $\square$

3.5. Remark. Let $\mathfrak{F}$ and $\mathfrak{H} = LF(H)$ be two saturated formations with H the canonical definition. From 1.5, 1.6 and 3.3 follow:

$$\mathfrak{F} \ll \mathfrak{H} \quad \Rightarrow \quad \mathfrak{F} \ll_{\mathsf{G}} \mathfrak{H} \quad \Leftarrow \quad \mathfrak{F} \ll_{\mathsf{D}} \mathfrak{H}.$$

3.6. Theorem. *Let $\mathfrak{H}$ be a saturated formations with canonical definition H and $\mathfrak{N} \subseteq \mathfrak{H}$. Let $p \in \mathbb{P}$. Is $\mathfrak{S}_{p'} \ll_{\mathsf{D}} \mathfrak{H}$, then $H(p) \subseteq H(q)$ for all $q \in \mathbb{P}$.*

Proof. Suppose there exists a group $G \in H(p) \setminus H(q)$. Define $W := C_p \wr_{\mathrm{reg}} G$. Then $W \in H(p)$ and therefore $W \in \mathfrak{H}$. Form hypothesis we have $W^{H(q)} \leqslant E$ for all $q \in \mathbb{P}$, where E is an $S_{p'}$-projector of W. Then hold $W^{H(q)} \leqslant O_{p'}(W)$ for all $q \in \mathbb{P}$. From [5, A (18.8)] follow that $O_{p'}(W) = 1$ and this proves that $W \in H(q)$ for all $q \in \mathbb{P}$. Then $G \in \mathsf{Q}(W) \subseteq H(q)$, contrary to the choice of G and so $H(p) \subseteq H(q)$ for all $q \in \mathbb{P}$. $\square$

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